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The unsteady expansion of a gas into a non-uniform near vacuum

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The paper generalizes an earlier problem of Grundy (1972) by considering the expansion of a (uniform) initially contained gas into a low-density non-uniform ambient atmosphere of density $\rho_0 r^{-k}$, where $k > 0$ and r is a non-dimensional radial co-ordinate. Regarding the flow as a perturbation of the perfect-vacuum expansion, we set up a boundary-value problem with boundary conditions on the contact front separating the two gases and on the strong shock which propagates into the ambient atmosphere. A large time solution to the problem can be developed by constructing an outer expansion valid near the contact front and an inner expansion valid near the shock. The matching process encounters two kinds of difficulty both of which imply that the large time solution is indeterminate from an asymptotic analysis alone.

The asymptotic analysis does show however that the shock velocity tends to a constant only for restricted values of k. For the remaining values the shock has a k-dependent power-law behaviour. The paper examines the location of the transition and determines the asymptotic power-law dependence of the shock velocity.

1. Introduction

In a previous paper (Grundy 1972, hereafter referred to as I), the author deals with the problem of how the unsteady expansion of an initially uniform mass of gas into a uniform ambient atmosphere degenerates into a perfect-vacuum expansion. In addition the paper gives an indication of how the large time solution of the resulting boundary-value problem could be attacked. In the present paper we extend these ideas to cover the expansion of an initially uniform gas into a non-uniform ambient atmosphere. This generalization is of importance in certain astrophysical problems (e.g. Parker 1963) concerning the propagation of flare-driven shocks in an ambient stellar wind.

The basic set-up for the problem has been indicated in I; the difference now is that we consider the ambient gas density to have the form $\rho_0 r^{-k}$, $k > 0$, where r is a non-dimensional radial co-ordinate and ρ_0 is the initial ambient density at $r = 1$. The problem can be reduced to solving the equations of gasdynamics subject to a condition on velocity along the contact front between the source gas and the ambient gas and the Rankine–Hugoniot conditions at the strong shock. By suitably scaling the variables, three parameters appear in the problem: k, σ (the geometry index) and γ_0.

The problem is initially attacked using the particle-path co-ordinate as one of the independent variables. As indicated in I, a uniformly valid large time solution, in which for mathematical and physical reasons we are primarily interested, consists of two expansions: an outer expansion valid near the contact front must be matched with an inner expansion which takes into account the boundary conditions at the shock. For certain values of k, those for which the asymptotic shock velocity is constant, matching difficulties are encountered which occur in other branches of gasdynamics, namely hypersonic small disturbance theory (Freeman 1965; Stewartson & Thompson 1968, 1970; Ellinwood 1967). We show in an appendix that the large time solution cannot be found by asymptotic analysis alone and that eigenfunctions appear in the higher-order terms which can be fully determined only by conditions at finite times.

What we can determine, however, is the asymptotic shock velocity, numerically as a function of σ, γ_0 and k. The important question of the existence of such an asymptotic solution is then examined. To do this we see that the zeroth-order term in the inner expansion is, in physical co-ordinates, the well-known similarity or progressing-wave solution (e.g. Courant & Friedrichs 1948; Sedov 1959). Certain hitherto undiscovered properties of these solutions are relevant in this context, and for this reason we examine the phase plane of these solutions. From such an examination it is apparent that, for the contact front and the shock to have the same asymptotic path, k has to be less than some critical value $k_c(\sigma, \gamma_0)$. For values of k greater than this the only possibility is a power-law behaviour of the shock with the index a function of k, σ and γ_0. The other parameter which fixes the shock path is apparently not determinate by the asymptotic analysis. In this case we examine the outer solution in some detail and we find that to first order it has the structure of the 'inertia-dominated' vacuum solution (Grundy 1969*a*, *b*). The basic feature is that it is impossible to close the set of equations which determines the asymptotic expansion and consequently it possesses an inherent indeterminacy.

After setting up the boundary-value problem, we divide the paper into two main parts: (i) the case $k \leqslant k_c$, where the ideas are in essence the same as I, and (ii) the case $k > k_c$. The main aim in (i) is to show the non-existence of solutions for $k > k_c$, to calculate k_c as a function of σ and γ_0 and to find the asymptotic shock speed as a function of σ, k and γ_0. In (ii) we treat the matching in a little more detail but our main concern is to determine the parameters which fix the asymptotic shock path.

2. Equations and boundary conditions

Adopting the same notation and non-dimensionalization as in I, the equations governing the problem can be written as

$$\frac{\partial(\rho r^\sigma)}{\partial t}+\frac{\partial(\rho u r^\sigma)}{\partial r} = 0, \tag{2.1a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{2.1b}$$

$$\frac{\partial(p/\rho^{\gamma_0})}{\partial t}+u\frac{\partial(p/\rho^{\gamma_0})}{\partial r} = 0 \tag{2.1c}$$

together with the contact-front boundary condition

$$u = \frac{2}{\gamma_4-1} \quad \text{on} \quad r = \frac{2t}{\gamma_4-1}+1. \tag{2.2}$$

Here γ_4 is the specific heat of the source gas. In the present case the ambient density is of the form

$$\rho = r^{-k}, \quad k > 0,$$

and we then have in the vacuum limit the strong shock relations

$$u_1 = \frac{2V}{\gamma_0+1}, \quad \rho_1 = \frac{r^{-k}(\gamma_0+1)}{\gamma_0-1}, \quad p_1 = \frac{2V^2 r^{-k}}{\gamma_0+1}. \tag{2.3}$$

For further details of the limiting procedure, the reader is referred to I and McLaughlin (1975). The problem is to solve (2.1) and find V subject to (2.2), (2.3) and the initial conditions $u = 0$, $\rho = r^{-k}$ and $p = 0$ for $r > 1$.

The aim is to develop a large time solution to the problem. To facilitate this it is, at first, easier to introduce the particle-path function ψ defined, from the continuity equation, by

$$\partial\psi/\partial r = \rho r^{\sigma}, \quad \partial\psi/\partial t = -\rho u r^{\sigma}. \tag{2.4a, b}$$

The equation of the shock in r, ψ space is obtained by integrating (2.4a) along the front of the shock, where $\rho = r^{-k}$, and invoking mass continuity through the shock. This procedure gives

$$r = \begin{cases} \{(\sigma-k+1)\psi+1\}^{1/(\sigma-k+1)}, & \sigma+1 \neq k, \\ e^{\psi}, & \sigma+1 = k, \end{cases} \tag{2.5}$$

as the required equation. In the first case we simplify by putting

$$(\sigma-k+1)\psi+1 = \phi \tag{2.6}$$

and a further change of variables is made by replacing

$$u \text{ by } \frac{2u}{\gamma_4-1}, \quad V \text{ by } \frac{\gamma_0+1}{\gamma_4-1}V, \quad \rho \text{ by } \frac{\gamma_0+1}{\gamma_0-1}\rho, \quad p \text{ by } \frac{\gamma_0+1}{(\gamma_4-1)^2}p.$$

The equations and boundary conditions then become, for $\sigma+1 \neq k$,

$$\frac{\partial(\rho u r^{\sigma})}{\partial r} + \rho^2 r^{2\sigma}\frac{\gamma_0+1}{\gamma_0-1}(\sigma-k+1)\frac{\partial u}{\partial\phi} = 0, \tag{2.7a}$$

$$\rho u\frac{\partial u}{\partial r} + \frac{(\gamma_0-1)}{2}\frac{\partial p}{\partial r} + \rho r^{\sigma}\frac{\gamma_0+1}{2}(\sigma+1-k)\frac{\partial p}{\partial\phi} = 0, \tag{2.7b}$$

$$p/\rho^{\gamma_0} = g(\phi) \tag{2.7c}$$

subject to the contact-front condition

$$u = 1 \quad \text{on} \quad \phi = 1 \tag{2.8}$$

and the shock conditions

$$\rho = r^{-k}, \quad p = r^{-k}V^2, \quad u = V \quad \text{on} \quad r = \phi^{1/(\sigma-k+1)}, \tag{2.9}$$

where $V = V(r)$.

For $\sigma+1 = k$ we have

$$\frac{\partial(\rho u r^{\sigma})}{\partial r} + \frac{\gamma_0+1}{\gamma_0-1}\rho^2 r^{2\sigma}\frac{\partial u}{\partial\psi} = 0, \tag{2.10a}$$

$$\rho u\frac{\partial u}{\partial r} + \frac{(\gamma_0-1)\partial p}{2\ \partial r} + \frac{\gamma_0+1}{2}\rho r^{\sigma}\frac{\partial p}{\partial\psi} = 0, \tag{2.10b}$$

$$p/\rho^{\gamma_0} = f(\psi) \tag{2.10c}$$

subject to $u = 1$ on $\psi = 0$ and the shock conditions (2.3) on $r = e^{\psi}$.

The functions $g(\phi)$ and $f(\psi)$ are found by evaluating p/ρ^{γ_0} at the rear of the shock, i.e.

$$g(\phi) = [V(\phi^{1/(\sigma-k+1)})]^2 \frac{k(\gamma_0-1)}{\phi^{\sigma-k+1}}, \quad f(\psi) = [V(e^{\psi})]^2 \exp\,[k(\gamma_0-1)\psi]. \tag{2.11}$$

3. The asymptotic solution

3.1. *Inner and outer expansions for large times*

The aim is to find asymptotic solutions of (2.7) and (2.10) for large r. The first step in such a procedure is to make an assumption about the shock velocity. It seems reasonable initially to take

$$V(r) = a_0\left(1+\sum_{n=1}^{\infty} b_n r^{\beta_n}\right) \quad \text{as} \quad r\to\infty, \quad \operatorname{Re}\beta_i < 0, \tag{3.1}$$

for then the shock would approach constant velocity for large times as does the contact front. Assumption (3.1) is fundamental and if the asymptotic theory which results is not consistent then we must re-examine it.

In (2.7) we put (replacing γ_0 by γ)

$$\rho = r^{-k/\gamma}R, \quad p = r^{-k}\Pi, \tag{3.2}$$

which gives

$$\frac{\partial}{\partial r}(Rur^{\sigma-k/\gamma}) + R^2 r^{2(\sigma-k/\gamma)}(\sigma-k+1)\frac{\gamma+1}{\gamma-1}\frac{\partial u}{\partial\phi} = 0, \tag{3.3a}$$

$$Rur^{-k/\gamma}\frac{\partial u}{\partial r} + \frac{\gamma-1}{2}\left(r^{-k}\frac{\partial\Pi}{\partial r} - kr^{-k-1}\Pi\right) + (\sigma-k+1)\frac{\gamma+1}{2}r^{\sigma-k-k/\gamma}$$
$$+(\sigma-k+1)\frac{\gamma+1}{2}r^{\sigma-k-k/\gamma}\frac{R\,\partial\Pi}{\partial\phi} = 0 \tag{3.3b}$$

and

$$\Pi/R^{\gamma} = g(\phi). \tag{3.3c}$$

For reasons which will become apparent, k, σ parameter space is divided into four different regimes, which we consider in turn.

Case 1: $0 < k/(\sigma+1) < 1$. We construct asymptotic expansions of R, u and Π for $r\to\infty$ with $\phi = O(1)$; these are termed the outer expansions, with u satisfying the boundary condition (2.8). The structure of the equations indicates that the only possible form for the outer expansion up to and including terms $O(r^{-(\sigma+1-k/\gamma)})$ is

$$\begin{aligned} u &= 1 + u_2(\phi)/r^{\sigma+1-k/\gamma} + \ldots, \\ \Pi &= \Pi_0(\phi) + \Pi_1(\phi)\,h(r) + \Pi_2(\phi)/r^{\sigma+1-k/\gamma} + \ldots, \\ R &= R_0(\phi) + R_1(\phi)\,h(r) + R_2(\phi)/r^{\sigma+1-k/\gamma} + \ldots, \end{aligned} \tag{3.4}$$

where the terms involving $h(r)$ are generic terms of a sequence of functions such that $h(r)\to 0$ as $r\to\infty$. It turns out that it is necessary to include only one such function for first-order matching. In (3.4)

$$u_2 = \frac{(k-\gamma\sigma)(\gamma-1)}{\gamma(\sigma-k+1)(\gamma+1)}\int_1^{\phi}\{R_0(y)\}^{-1}dy, \tag{3.5a}$$

satisfying $u_2 = 0$ on $\phi = 1$,

$$\Pi_0 = \alpha_0, \quad \Pi_1 = \alpha_1, \quad \Pi_2 = \alpha_2 - \frac{k\gamma}{\sigma\gamma-k}u_2, \tag{3.5b}$$

$$R_0 = \left(\frac{\alpha_0}{g(\phi)}\right)^{1/\gamma}, \quad R_1 = \frac{\alpha_1 R_0}{\gamma\alpha_0}, \quad R_2 = \frac{R_0\Pi_2}{\gamma\alpha_0}, \tag{3.5c}$$

where the α's are constants. Clearly, in order to find U_2, Π_2 and R_2 explicitly, we should need to know the shock velocity $V(r)$ for all r; thus the outer expansion is in this sense indeterminate. However, by making the assumption (3.1) we may find the behaviour of (3.5) as $\phi \to \infty$:

$$R_0(\phi) = \left(\frac{\alpha_2}{a_0^2}\right)^{1/\gamma} \phi^{-k(\gamma-1)/\sigma-k+1}\left(1-\frac{2b_1}{\gamma}\phi^{\beta_1/(\sigma-k+1)}+\ldots\right), \tag{3.6a}$$

$$u_2(\phi) = \frac{(k-\gamma\sigma)(\gamma-1)}{\gamma(\gamma+1)}\left(\frac{a_0^2}{\alpha_0}\right)^{1/\gamma}\left\{\frac{\phi^{(\sigma+1-k/\gamma)/\sigma+1-k}}{\sigma+1-k/\gamma}+\frac{2b_1\phi^{(\sigma+1-k/\gamma+\beta_1)/\sigma+1-k}}{\gamma(\sigma+1)-k+\gamma\beta_1}+I^*\ldots\right\}, \tag{3.6b}$$

with a corresponding expression for Π_2. Here I^* is the finite part of the integral for u_2. For values of β_1 given by $\beta_1 = -n(\sigma+1-k/\gamma)$, $n = 1, 2, \ldots$, logarithms are introduced into the expansions for u_2 and Π_2.

Now the shock is given by $\phi = r^{\sigma+1-k}$, and it is clear that the outer expansions (3.4) cannot satisfy the boundary conditions there. In fact an examination of (3.4) with (3.6) reveals that the outer expansion is not uniformly valid when $\phi = O(r^{\sigma+1-k})$. In order to look at this inner region we choose an inner variable

$$\phi_1 = \phi r^{k-\sigma-1} \tag{3.7}$$

so that $\phi_1 = 1$ on the shock.

Rewriting (2.7) in terms of ϕ_1 and putting $p = r^{-k}P$ and $\rho = r^{-k}S$, we have

$$r\frac{\partial}{\partial r}(Su)+(\sigma-k)Su-(\sigma-k+1)\phi\frac{\partial(Su)}{\partial\phi_1}+\frac{\gamma+1}{\gamma-1}(\sigma-k+1)S^2\frac{\partial u}{\partial\phi_1} = 0, \tag{3.8a}$$

$$Sur\frac{\partial u}{\partial r}-(\sigma-k+1)Su\,\phi_1\frac{\partial u}{\partial\phi_1}+\frac{\gamma-1}{2}\left(r\frac{\partial P}{\partial r}-kP\right) -\frac{\gamma-1}{2}(\sigma+1-k)\phi_1\frac{\partial P}{\partial\phi_1}+\frac{\gamma+1}{2}(\sigma-k+1)S\frac{\partial P}{\partial\phi_1} = 0 \tag{3.8b}$$

and

$$PS^{-\gamma} = \phi_1{}^{k(\gamma-1)/\sigma-k+1}\,V^2(r\phi_1^{1/(\sigma-k+1)}). \tag{3.8c}$$

The shock relations suggest that as we have expanded $V(r)$ in the form (3.1) we must expand P as

$$P = P_0(\phi_1)+\sum_{n=1}^{\infty} b_n P_n(\phi_1)\,r^{\beta_n}, \tag{3.9}$$

with corresponding expansions for u and S. These are the inner expansions. If we substitute (3.9) and (3.1) into (3.8) we find that the zeroth-order terms satisfy

$$(\sigma-k)S_0U_0-(\sigma-k+1)\phi_1(S_0U_0)'+\frac{\gamma+1}{\gamma-1}(\sigma-k+1)S_0^2U_0' = 0, \tag{3.10a}$$

$$(\sigma-k+1)S_0U_0\phi_1U_0'+k\frac{\gamma-1}{2}P_0+\frac{\gamma-1}{2}(\sigma+1-k)\phi_1P_0'+\frac{\gamma+1}{2}(\sigma-k+1)S_0P_0' = 0, \tag{3.10b}$$

$$P_0 = a_0^2\phi_1^{k(\gamma-1)/\sigma+1-k}S_0^\gamma, \tag{3.10c}$$

the boundary conditions on the shock being

$$S(1) = 1, \quad U(1) = a, \quad P(1) = a. \tag{3.10d}$$

Before going on with the solution of the zeroth-order inner problem we make some brief remarks about the matching of the two expansions. The procedure has striking similarities with those encountered in hypersonic small disturbance theory (e.g. Freeman 1965). The actual mathematical difficulties involved were highlighted by Ellinwood (1967) and more fully investigated by Stewartson & Thompson (1968, 1970). Although our problem is different the underlying ideas are the same so we relegate the details to the appendix.

Case 2: $k = 0$. It is clear from a glance at the equation (3.5*b*) for Π_2 that the term involving u_2 vanishes when $k = 0$. So in this case the outer expansions take on a different form; essentially, in the notation of (3.4) the error terms $O(1/r^{\sigma+1-k/\gamma})$ do not exist. Without going through the details of the matching, which is essentially the same as in case 1, we do indicate the modifications to the outer expansions. The consequent modifications to the matching procedures and the inner expansions are not difficult to work out (see appendix).

The outer expansions for u and Π are

$$u = 1+u_2(\phi)/r^{\sigma+1}+\ldots, \tag{3.11a}$$

$$\Pi = \Pi_0(\phi)+h(r)\,\Pi_1(\phi)+\Pi_2(\phi)/r^{2(\sigma+1)}+\ldots, \tag{3.11b}$$

where $h(r)$ has the same interpretation as in case 1. Here

$$u_2 = \frac{-\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)}\int_1^{\phi}\{R_0(y)\}^{-1}\,dy, \tag{3.12a}$$

$$\Pi_0 = \alpha_0, \quad \Pi_1 = \alpha_1, \quad \Pi_2 = \alpha_2+\frac{2}{(\gamma+1)}\int_1^{\phi}u_2(y)\,dy. \tag{3.12b}$$

The rest of the asymptotic solution follows as before except that, owing to the structure of the error term $\Pi_2(\phi)$, the case $\beta_1 = -2(\sigma+1)$ may be included. In that event we choose $h(r) = (\log r)/r^{\sigma+1}$ and choose α_1 such that the spurious terms which arise when (3.11*b*) is written in inner variables and which would otherwise cause difficulty may be removed. Again we must exclude $\beta_1 = -(\sigma+1)$ and conclude that $I^* = 0$ (see appendix).

Case 3: $k/(\sigma+1) = 1$. In this case we have a different error term in the outer expansion for Π. Specifically, using (2.10) we expand

$$u = 1+u_2(\phi)/r^{k(\gamma-1)/\gamma}+\ldots, \tag{3.13a}$$

$$\Pi = \Pi_0+h(r)\,\Pi_1+\Pi_2(\phi)/r^{k(\gamma-1)/\gamma}+\ldots, \tag{3.13b}$$

with

$$u_2 = \left(\frac{k}{\gamma}+1-k\right)\frac{(\gamma-1)}{(\gamma+1)}\int_0^{\psi}\{R_0(y)\}^{-1}\,dy,$$

$$\Pi_0 = \alpha_0, \quad \Pi_1 = \alpha_1, \quad \Pi_2 = \frac{2k(\gamma-1)}{\gamma(\gamma+1)}\int_0^{\psi}u_2(y)\,dy+\frac{k\alpha_0 u_2}{1+k/\gamma-k}+\alpha_2.$$

We now follow the same procedure as in case 1 except that we use the inner variable

$$\psi_1 = e^{\psi}/r. \tag{3.14}$$

Again we conclude that β_1 cannot be equal to $-k(\gamma-1)/\gamma$ and that $I^* = 0$. In addition, for the same reason the constant α_2 is chosen such that, when (3.13*b*) is written in inner variables, the term involving the finite part of the integral occurring in the expression for Π_2 vanishes (see appendix).

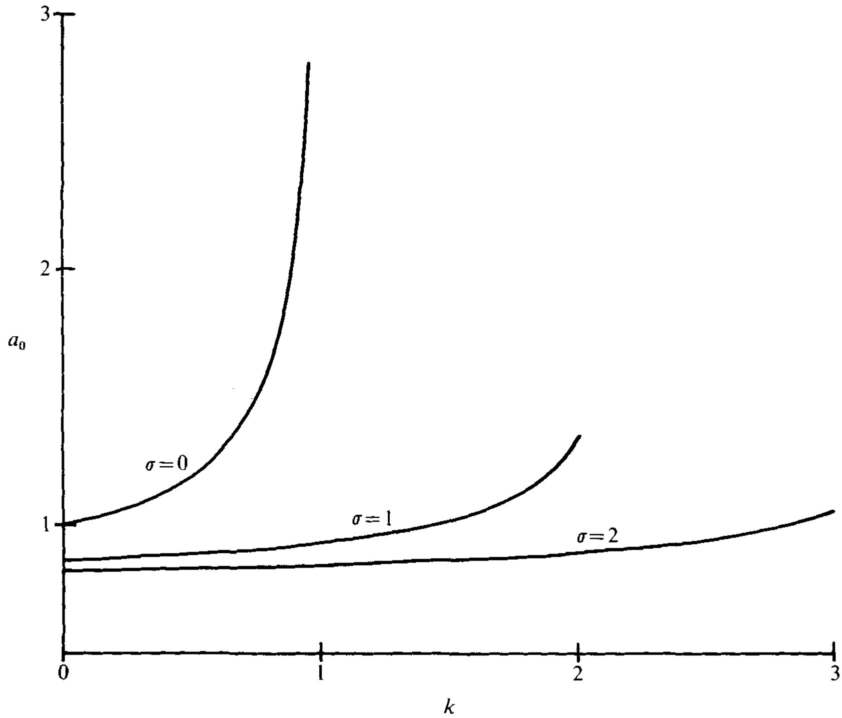

FIGURE 1. a_0 for $\gamma = \frac{5}{3}$.

Case 4: $1 < k/(\sigma+1) < 2\gamma/(\gamma+1)$. This case is characterized by another type of error in the outer expansion for Π. We expand

$$u = 1 + u_2(\phi)/r^{\sigma+1-k/\gamma} + \ldots, \tag{3.15}$$

$$\Pi = \Pi_0(\phi) + h(r)\,\Pi_1(\phi) + \Pi_2(\phi)/r^{2(\sigma+1)-k(\gamma+1)/\gamma} + \ldots,$$

where

$$\Pi_0 = \alpha_0, \quad \Pi_1 = \alpha_1, \quad \Pi_2 = \frac{\sigma+1-k/\gamma}{\sigma+1-k}\frac{2}{\gamma+1}\int_1^\phi u_2(y)\,dy + \alpha_2$$

and

$$u_2 = \frac{(k/\gamma-\sigma)(\gamma-1)}{(\sigma+1-k)(\gamma+1)}\int_1^\phi \{R_0(y)\}^{-1}\,dy. \tag{3.16}$$

Because $\sigma+1-k$ is now negative, the inner limit is $\phi \to 0$ and the outer limit is $\phi_1 \to \infty$. Again we have virtually the same matching process as in case 3. However, for $\beta_1 = 2(\sigma+1) - k(\gamma+1)/\gamma$, $h(r)$ is put equal to $(\log r)/r^{2(\sigma+1)-k(\gamma+1/\gamma)}$ with the appropriate value of α in order to get rid of a logarithmic term occurring in the inner limit of the outer expansion. α_2 is now obtained in terms of p^* and the finite part of the integral in the expression for Π_2 (see appendix).

3.2. *The existence of the inner zeroth-order solution*

We now concentrate on the discussion of the solution of (3.10). If a solution exists it determines the zeroth-order shock speed a_0 as a function of k, σ and γ; *this is displayed for* $\gamma = \frac{5}{3}$ *in figure 1.* It is soon apparent, however, from the numerical integrations of

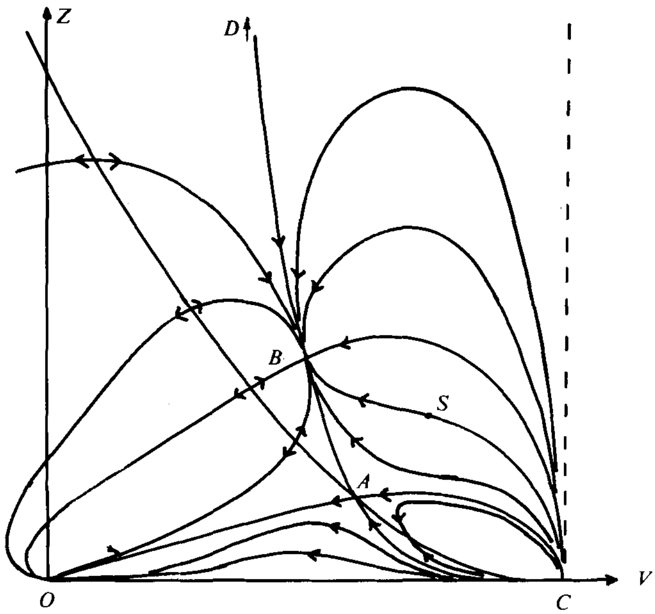

FIGURE 2. Integral curves for $\delta = 1$, $k < \sigma\gamma$.

(3.10) that the existence of solutions must be called into question in certain regions of parameter space. We therefore look at the question of existence in a little more detail.

For this purpose it is easier to cast (3.10) into a more 'physical' form. We note that the zeroth-order solutions for U_0, P_0 and S_0 are equivalent to the well-known similarity solutions of one-dimensional gasdynamics (see, for example, Sedov 1959, chap. IV). Although Sedov gives a detailed account of the solutions in a restricted region of k, σ parameter space, not enough information is available for the problem under discussion.

Following Sedov we write

$$u = \delta(r/t)\, V(\lambda), \quad a^2 = \delta^2(r/t)^2 Z(\lambda), \tag{3.17), (3.18}$$

where $a^2 = \gamma p/\rho$ and $\lambda = rt^{-\delta}$ is the similarity variable. Substituting into the equations of gasdynamics (2.1), with $\gamma = \gamma_0$, we may reduce the equations to a single first-order equation in Z and V, namely

$$dZ/dV = ZS(V,Z)/(1-V)\,Q(V,Z),$$

where

$$S = \{2(V-1/\delta)+(\gamma-1)(\sigma+1)\,V\}(1-V)^2+(\gamma-1)\,V(V-1/\delta)(1-V) \\ -Z[2(V-1/\delta)+(K/\delta)(\gamma-1)], \tag{3.19a}$$

$$Q = V(V-1/\delta)(1-V)+Z\{(\sigma+1)\,V-K/\delta\} \tag{3.19b}$$

and

$$K = 2+\delta(k-2)/\gamma. \tag{3.19c}$$

The position of the strong shock is given by

$$Z = Z_s = 2\gamma(\gamma-1)/(\gamma+1)^2, \quad V = V_s = 2/(\gamma+1). \tag{3.20}$$

Equations (3.19) and (3.20), with $\delta = 1$, are equivalent to (3.10), and we now proceed to examine the existence of the solutions.

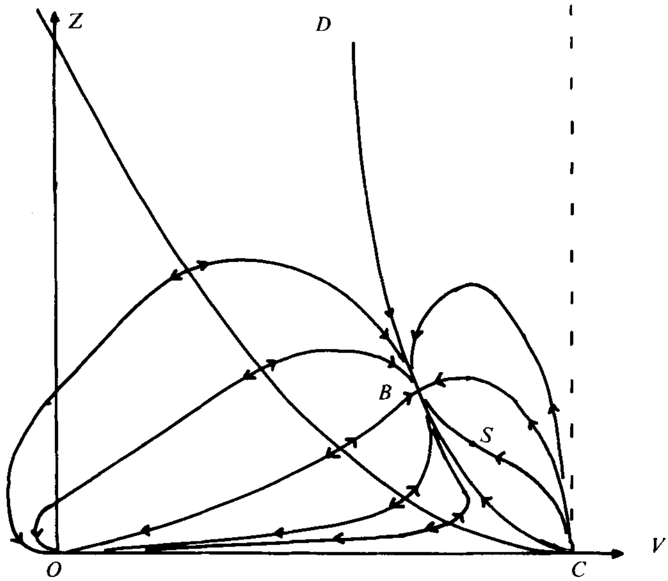

FIGURE 3. Integral curves for $\delta = 1$, $k > \sigma\gamma$.

With $\delta = 1$, $\lambda = rt^{-1}$ and hence it can be shown that the contact front is given by

$$V = 1, \tag{3.21}$$

where, of course, $\lambda = 1$. In terms of the particle-path notation of (3.10) this is the limit $\phi_1 \to 0$.

A first step towards understanding the problem of the existence of solutions of (3.10) is to examine the phase plane of (3.19) with $\delta = 1$. For $k < \sigma\gamma$ the situation is shown in figure 2. The singular point C is a complicated one of the nodal type located at $Z_c = 0$, $V_c = 1$, where, in the notation of (3.10), $\phi_1 = 0$ or $\lambda = 1$. A is a saddle point located at $Z_A = (1-V_A)^2$ with $V_A = k/\gamma\sigma$. D is a saddle point at $Z_D = \infty$, $V_D = k/\gamma(\sigma+1)$. The final singular point of relevance is B, a node, located at

$$Z_B = 2(\sigma+1)^2 \bigg/ \left\{\sigma + \frac{\gamma+1}{\gamma-1}\right\} \left\{2(\sigma+1) - \frac{k}{\gamma}[(\gamma+1)+\sigma(\gamma-1)]\right\},$$

$$V_B = 2/\{(\gamma+1)+\sigma(\gamma-1)\}.$$

If $Z_B < 0$, then B lies in the lower half-plane and D becomes a node; the condition is that $k > 2(\sigma+1)/\{\sigma(\gamma-1)+(\gamma+1)\}$. The other relevant point in figure 2 is the location S of the strong shock, given by (3.20). The arrows in all phase-plane diagrams indicate the directions of increasing λ.

The zeroth-order solution of (3.10) is simply represented by the integral curve joining S to C. As k increases from zero, for fixed σ, the integral curve joining B (or D if $Z_B < 0$) and A moves to the right until, at a value $k = k_c$, S lies on it; it can be shown that B is always above S in this event. Clearly no integral curve passing through S can possibly reach C for k greater than this critical value. The situation is essentially the same for $k \geqslant \sigma\gamma$ in figure 3. Here $V_A \geqslant 1$, with A coinciding with C at equality. Again, as k increases for fixed σ the integral curve joining C, B and D, or C and D if $Z_B < 0$, moves to the right until S lies on it when $k = k_c$. For values of $k > k_c$ no

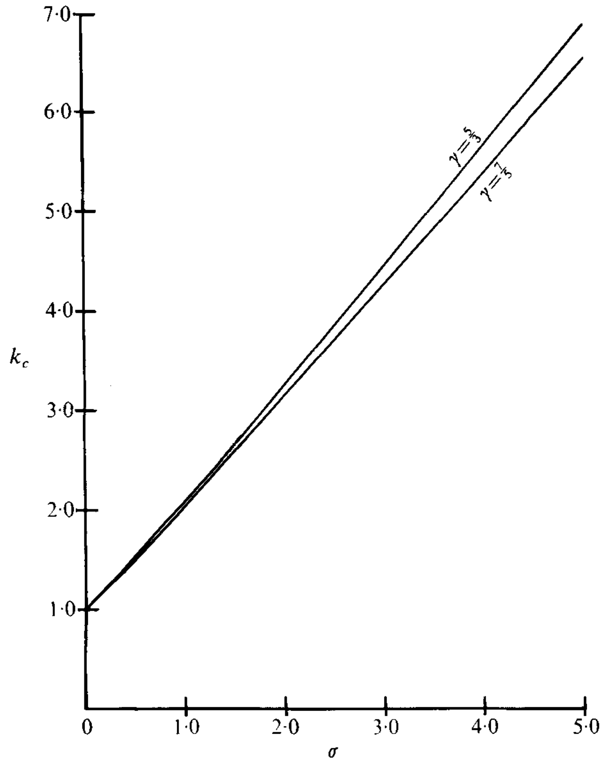

FIGURE 4. k_c *vs.* σ for $\gamma = \frac{5}{3}$ and $\frac{7}{5}$.

integral curve can possibly join S and C. So we can conclude that solutions of (3.10) exist only for $k < k_c$. For a detailed derivation of the nature and location of the singular points we refer the interested reader to McLaughlin (1975).

The behaviour of k_c as a function of σ for $\gamma = \frac{5}{3}$ and $\frac{7}{5}$ is shown in figure 4. This is calculated numerically by obtaining *a priori* bounds on k_c which are refined by an appropriate iteration scheme. Basically, if k_1 and k_2 are, for a particular σ, upper and lower bounds for k_c, then we bisect the interval and form $k_3 = \frac{1}{2}(k_1+k_2)$; we now integrate with $k = k_3$ from A (or C if $k > \gamma\sigma$) along the integral curve joining the singular points until $Z = Z_s$, $V = V_1$. If $V_1 < V_s$, k_3 is too small and we replace k_2 by k_3; on the other hand, if $V_1 > V_s$, k_3 is too large and we replace k_1 by k_3. This algorithm is repeated until we have sufficiently fine bounds on k_c.

We now turn to the phase-plane analysis for $k > k_c$. As we shall show in §3.3, in this case we need to choose $\delta > 1$. Sedov (1959) does not deal with this situation so we must discuss this case in some detail. In figure 5 we show a typical phase-plane diagram for $\delta > 1$; for this particular case $K = \{2+\delta(k-2)\}/\gamma < \sigma+1$, the only difference for other values of K being that, for $K = \sigma+1$, F lies on $Z = 0$ but directly below G while for $K > \sigma+1$ it lies to the left of G on $Z = 0$; the essential topography is, however, the same. We shall explain the location S of the strong shock in a moment.

The first observation to make is that, since dr/dt is always finite on the contact front, for $\delta > 1$ it is located at $\lambda = 0$, which, as we shall see in §3.3, is the outer limit of the inner solution for $\delta > 1$. It turns out that the only possible point which can

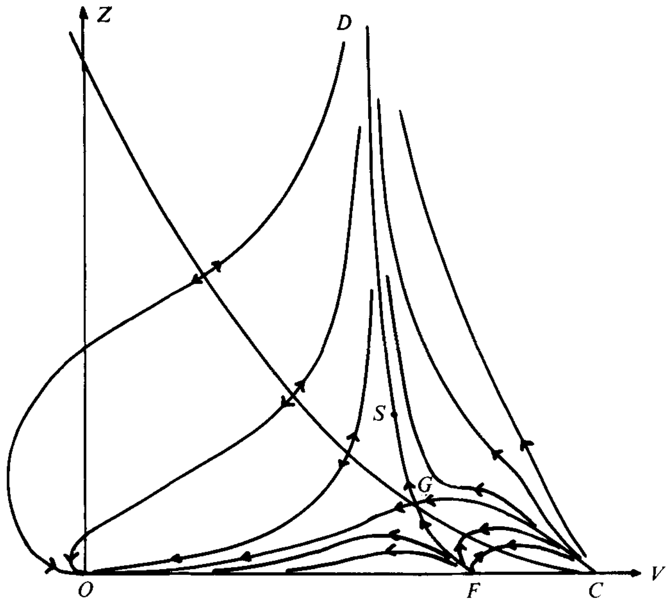

FIGURE 5. Integral curves for $\delta > 1$.

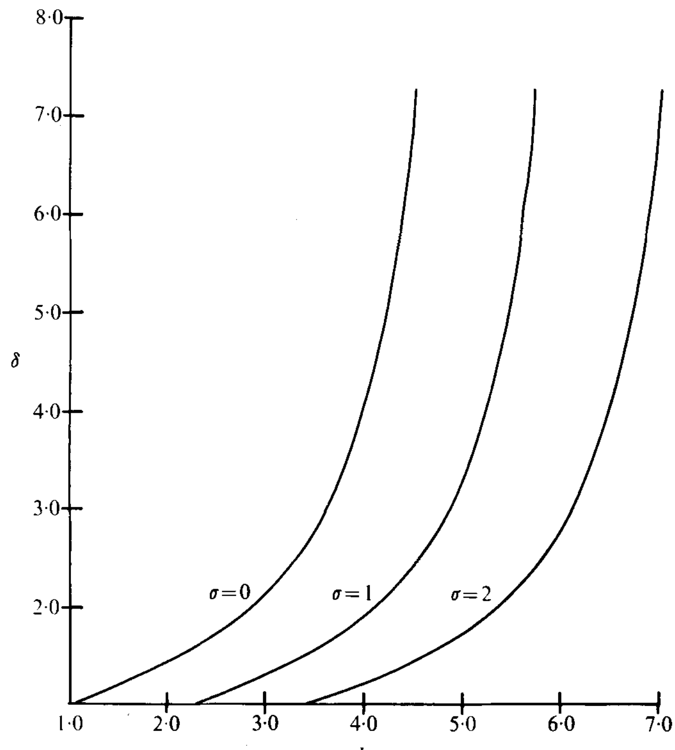

FIGURE 6. δ for $\gamma = \frac{5}{3}$.

represent this limit in figure 5 is F, located at $V_F = 1/\delta$, $Z_F = 0$. The remaining relevant singular points are C, at $V_C = 1$, $Z_C = 0$, the saddle point G, at

$$V_G = \frac{1}{2}\left(\frac{K}{\delta\sigma}+\frac{\delta-1}{\delta\sigma}+1\right)-\left[\frac{1}{4}\left(\frac{K}{\delta\sigma}+\frac{(\delta-1)}{\delta\sigma}+1\right)^2-\frac{K}{\delta\sigma}\right]^{\frac{1}{2}},$$

$$Z_G = (1-V_G)^2,$$

and finally the node D, at $V_D = K/\delta(\sigma+1)$, $Z_D = \infty$. Clearly the only possible way to get from S to F is for S to lie on the integral curve Δ passing through D, G, and F. For each value of K there exists a δ which makes S lie on this curve, hence δ is now a function of K, γ and σ, and may be found numerically.

Clearly $\delta \to 1$ as $k \to k_c(\gamma, \sigma)$ from above, but there is little hope of obtaining δ analytically except in one particular case. For $K = 2+\delta(k-2)/\gamma = \sigma+1$, the curve Δ is given by $V = 1/\delta$. The condition that Δ passes through S gives

$$\delta = \delta^* = \tfrac{1}{2}(\gamma+1),$$

which, together with $K = \sigma+1$, implies

$$k = k^* = 2\{\gamma(\sigma+1)+(\gamma-1)\}/(\gamma+1).$$

For other values of k we must resort to some numerical technique for the evaluation of δ. Essentially the method is the same as that used to evaluate k_c; for further details the reader is referred to McLaughlin (1975). The variation of δ with k for $\gamma = \frac{5}{3}$ and $\sigma = 0$, 1 and 2 is shown in figure 6.

Having found δ for $k > k_c$ we now proceed to give a brief discussion of the asymptotic solution in this case.

3.3. *The asymptotic solution for $k > k_c$*

In a previous section we noted the non-existence of the inner solution for $k > k_c$ with $\delta = 1$. The question therefore arises of how to construct the asymptotic solution in this regime of k, σ parameter space. The basic assumption we must re-examine is the behaviour of the shock for large r, equation (3.1). There we assumed that the shock tended to a constant velocity for large r; however, now we must explore the possibility that this assumption is no longer correct. Instead we write

$$V = b_0 r^{(\delta-1)/\delta}\left\{1+\sum_{n=1}^{\infty} b_n r^{\alpha_n}\right\}, \quad \mathrm{Re}\,\alpha_n < 0. \tag{3.22}$$

We now derive the formal asymptotic solution. The assumption embodied in (3.22) for the shock velocity and the shock conditions themselves lead one to seek expansions of u, ρ and p of the following form:

$$u = b_0 r^{\epsilon}\{U_0(\phi_1)+b_1 U_1(\phi_1) r^{\alpha_1}+\ldots\}, \quad \epsilon = (\delta-1)/\delta, \tag{3.23a}$$

and
$$\rho = \left(\frac{\gamma+1}{\gamma-1}\right) r^{-k}\{\rho_0(\phi_1)+b_1\rho_1(\phi_1) r^{\alpha_1}+\ldots\}, \tag{3.23b}$$

with a corresponding expansion for p. These are termed the inner expansions valid in the limit $r \to \infty$ with $\phi_1 = O(1)$, ϕ_1 being given by (3.7). The strong shock conditions give

$$U_0(1) = \rho_0(1) = 1, \quad U_1(1) = 1, \quad \rho_1(1) = 0. \tag{3.24}$$

Making the appropriate change of variable, we substitute (3.23) into the full equations (2.7) and equate zeroth- and first-order terms to zero. The zeroth-order equations are

$$\phi_1[U_0\rho_0'+\rho_0 U_0'] - \left(\frac{\gamma+1}{\gamma-1}\right)\rho_0^2 U_0' + \frac{\sigma+\epsilon-k}{k-\sigma-1}\rho_0 U_0 = 0, \tag{3.25a}$$

$$\rho_0 U_0 \phi_1 U_0' + \frac{\epsilon\rho_0 U_0^2}{k-\sigma-1} + \frac{\gamma-1}{2}\phi_1 P_0' - \frac{\gamma+1}{2}\rho_0 P_0' - \frac{(\gamma-1)(k-2\epsilon)P_0}{2(k-\sigma-1)} = 0 \tag{3.25b}$$

and

$$P_0 = \rho_0^\gamma \phi_1^{-[k(\gamma-1)+2\epsilon]/(k-\sigma-1)}. \tag{3.25c}$$

Going to first order in (3.23) we obtain

$$\phi_1\{U_0\rho_1' + U_1\rho_0' + \rho_0 U_1' + \rho_1 U_0'\} - \left(\frac{\gamma+1}{\gamma-1}\right)\{\rho_0^2 U_1' + 2\rho_0\rho_1 U_0'\} + \frac{\sigma+\epsilon-k+\alpha_1}{k-\sigma-1}(\rho_0 U_1 + \rho_1 U_0) = 0, \tag{3.26a}$$

$$\phi_1\{\rho_0 U_0 U_1' + \rho_0 U_1 U_0' + \rho_1 U_0 U_0'\} + \frac{\alpha_1\rho_0 U_0 U_1}{k-\sigma-1} + \epsilon U_0\{\rho_1 U_0 + 2\rho_0 U_1\} + \frac{\gamma-1}{2}\phi_1 p_1' - \frac{\gamma+1}{2}\{\rho_0 p_1' + \rho_1 p_0'\} - \frac{(\gamma-1)}{2}\frac{(k-\alpha_1-2\epsilon)}{k-\sigma-1}P_1 = 0, \tag{3.26b}$$

$$P_1 = P_0\left\{\frac{\gamma\rho_1}{\rho_0} + 2\phi_1^{-\alpha_1/(k-\sigma-1)}\right\}. \tag{3.26c}$$

The expansions (3.23) are valid near the shock. To investigate their validity as $\phi_1 \to \infty$, i.e. as the outer region near the contact front is approached, we examine the solutions of (3.25) and (3.26) in this limit. We are aided in this task by the knowledge that the singular point F in the Sedov phase plane is in fact the outer limit of the inner solution. That being so, we may write down the zeroth-order behaviour of U_0 and ρ_0 near F in terms of the Sedov similarity variable λ; knowing λ as a function of ϕ_1 we may consequently find the zeroth-order behaviour of U_0 and ρ_0 in terms of ϕ_1. Proceeding from there in a more formal manner we can generate the appropriate series solutions for $U_0(\phi_1)$, $\rho_0(\phi_1)$ and $P_0(\phi_1)$ as $\phi_1 \to \infty$; these are

$$U_0 = A_0\phi_1^{-\epsilon/(k-\sigma-1)}\{1 + A_1\phi_1^{-(\gamma-1)(\sigma+1)/(k-\sigma-1)} + \ldots\}, \tag{3.27a}$$

$$\rho_0 = B_0\phi_1\{1 + B_1\phi_1^{-\beta_1/(k-\sigma-1)} + \ldots\}, \tag{3.27b}$$

with a corresponding expression for P_0. Here

$$A_1 = \frac{[\gamma(\sigma+1)(1-\epsilon) + 2\epsilon - k]B_0^{\gamma-1}}{2\epsilon(\sigma+1)A_0^2}, \quad B_0 = \frac{\gamma-1}{\epsilon(\gamma+1)},$$

$$\beta_1 = \min[1, (\gamma-1)(\sigma+1)]$$

and A_0 is undetermined by the above analysis. If $\beta_1 = 1$, then B_1 is also undetermined but can be found numerically if necessary, while if $\beta_1 = (\gamma-1)(\sigma+1)$ then

$$B_1 = \frac{-(\gamma+1)(1-\epsilon)[\gamma(\sigma+1)(1-\epsilon) + 2\epsilon - k]B_0^\gamma}{2\epsilon A_0^2[1-(\gamma-1)(\sigma+1)]}.$$

The equations for the first-order variables are linear and inhomogeneous and the solution can be split up into a particular integral and a complementary function which will include two arbitrary constants. We may then generate appropriate series solutions for the first-order variables; for U_1 and ρ_1 these can be written as

$$U_1 = \phi_1^{-(\alpha_1+\epsilon)/(k-\sigma-1)}\{U_1^{(1)}(\phi_1) + U_1^{(2)}(\phi_1)\}, \tag{3.28a}$$

$$\rho_1 = \phi_1^{1-\alpha_1/(k-\sigma-1)}\{S_1^{(1)}(\phi_1) + S_1^{(2)}(\phi_1)\}, \tag{3.28b}$$

where the leading terms in the expansions of the particular integrals $U_1^{(1)}$ and $S_1^{(1)}$ are

$$U_1^{(1)} = -\epsilon A_0 B_2/\alpha_1 + \ldots, \quad S_1^{(1)} = B_0 B_2 + \ldots, \tag{3.29}$$

while for the complementary functions $U_1^{(2)}$ and $S_1^{(2)}$ we have

$$U_1^{(2)} = -\epsilon A_0 B_4/\alpha_1 + A_0 A_5 \phi_1^{-\{1+(\gamma-1)(\sigma+1)\}/(k-\sigma-1)} + \ldots, \tag{3.30a}$$

$$S_1^{(2)} = B_0\{B_4 + B_5 \phi_1^{-1/(k-\sigma-1)} + \ldots\}. \tag{3.30b}$$

In (3.29) and (3.30) B_2 is indeterminate, B_4 and B_5 are independent arbitrary constants and A_5 is linearly dependent upon B_5.

It is clear from the behaviour of U_0 and U_1 as $\phi_1 \to \infty$ that expansions (3.23) break down when $\phi_1 = O(r^{k-\sigma-1})$ or $\phi = O(1)$. Writing (3.23) in terms of the outer variable ϕ and taking the limit $r \to \infty$ we find $u = O(1)$, $\rho = O(r^{-(\sigma+1)})$ and $p = O(r^{-\gamma(\sigma+1)})$. Guided by these indications we seek outer expansions of the form

$$u = b_0\left\{u_0(\phi) + \frac{u_1(\phi)}{r^\alpha} + \ldots\right\}, \tag{3.31a}$$

$$\rho = \frac{\gamma+1}{\gamma-1} r^{-(\sigma+1)}\left\{R_0(\phi) + \frac{R_1(\phi)}{r^\beta} + \ldots\right\}, \tag{3.31b}$$

with a corresponding expansion for p; here α and β are taken to be positive constants which are to be determined.

If we substitute (3.31) into (2.7*a*) we obtain

$$R_0 u_0 + (k-\sigma-1)\frac{\gamma+1}{\gamma-1} R_0^2 u_0' + \left\{\alpha R_0 u_1 + (k-\sigma-1)\frac{\gamma+1}{\gamma-1} R_0^2 u_1' + R_0 u_1\right\}\frac{1}{r^\alpha}$$
$$+\left\{\beta u_0 R_1 + (k-\sigma-1)\frac{\gamma+1}{\gamma-1} 2R_0 R_1 u_0' + R_1 u_0\right\}\frac{1}{r^\beta} + \ldots = 0. \tag{3.32}$$

The terms $O(1)$ give on integrating with $u_0(1) = 1/b_0$

$$\log(b_0 u_0) = \frac{-(\gamma-1)}{(\gamma+1)(k-\sigma-1)}\int_1^\phi \frac{dy}{R_0(y)}. \tag{3.33}$$

Now if $\alpha < \beta$ then terms $O(r^{-\alpha})$ give

$$\alpha R_0 u_1 + (k-\sigma-1)\frac{\gamma+1}{\gamma-1} R_0^2 u_1' + R_0 u_1 = 0,$$

which, using (3.33), gives

$$u_1 = C_1 u_0^{1+\alpha},$$

where C_1 is an arbitrary constant. The boundary condition on the contact front, however, implies $C_1 = 0$ and hence, if $\alpha < \beta$, $u_1 = 0$. If on the other hand we take $\alpha > \beta$, the $O(r^{-\beta})$ terms in (3.32) give a zero value for R_1 or, if $\beta = 1$, an arbitrary non-zero value. The other possibility is that $\alpha = \beta$. In this event (3.32) gives to first order

$$(k-\sigma-1)\frac{\gamma+1}{\gamma-1}R_0^2 u_1' + (\alpha-1)R_1 u_0 + (\alpha+1)R_0 u_1 = 0. \tag{3.34}$$

If we now insert (3.31) into (2.7*b*) we get

$$\left\{\frac{\gamma(\gamma-1)}{2}(\sigma+1)\Pi_0 + (k-\sigma-1)\frac{\gamma+1}{2}R_0\Pi_0'\right\}r^{-(\gamma-1)(\sigma+1)} + \alpha R_0 u_0 u_1 r^{-\alpha} + \dots = 0,$$

there being no term $O(r^{-\beta})$. Clearly, for a non-zero u_1, $\alpha \geqslant (\gamma-1)(\sigma+1)$. First, if $\alpha > (\gamma-1)(\sigma+1)$ then terms $O(r^{-(\gamma-1)(\sigma+1)})$ give, on integration,

$$\Pi_0 = C_0^*(b_0 u_0)^{\gamma(\sigma+1)},$$

where C_0^* is arbitrary. Second, if $\alpha = (\gamma-1)(\sigma+1)$ then

$$\Pi_0' + \frac{\gamma(\sigma+1)(\gamma-1)}{(\gamma+1)(k-\sigma-1)}\frac{\Pi_0}{R_0} = -\frac{2(\gamma-1)(\sigma+1)u_0 u_1}{(\gamma+1)(k-\sigma-1)}. \tag{3.35}$$

It will be noticed that if $\alpha = \beta = (\gamma-1)(\sigma+1)$ then (3.33)–(3.35) will determine the asymptotic expansion; unfortunately this is not a closed set of equations. Unless we relax the boundary condition on u_1 which imposed the condition $\alpha \geqslant \beta$ or alternatively demand that $\alpha > (\gamma-1)(\sigma+1)$, the difficulty is inescapable. Taking the first alternative would require inserting another layer between the inner layer and the contact front. However, if we do this it soon becomes clear that matching with the inner solution is impossible. We may rule out the second alternative on two counts. First, there must be a term $O(r^{-(\gamma-1)(\sigma+1)})$ in the expansion for u so that it will match with the corresponding term in the inner solution. The second reason for rejection is that we cannot match the pressure expansion if $\alpha \neq (\gamma-1)(\sigma+1)$. So we must conclude that the asymptotic expansion is indeterminate in the sense that the hierarchy of differential equations for the coefficients does not form a closed set.

It is important and interesting to note that this difficulty occurs in the inviscid theory describing the expansion of a gas into a vacuum. For that problem it was pointed out by Grundy (1969*a*) that to zeroth order in a large time expansion the velocity is an arbitrary function of the particle-path co-ordinate; this is the so-called 'inertia-dominated' regime. This regime, together with its inherent arbitrary nature, also occurs in the problem of a steady axisymmetric jet expanding into a vacuum (Grundy 1968). For a further discussion of this point we refer the reader to McLaughlin (1975).

Although we cannot obtain an explicit asymptotic solution to the outer problem we can make some progress with the matching procedure. First of all let us assume expansions for $u_0(\phi)$ and $R_0(\phi)$ of the form

$$u_0(\phi) = A_0^*\phi^{\alpha_0^*}\{1 + A_1^*\phi^{\alpha_1^*} + \dots\}, \tag{3.36a}$$

$$R_0(\phi) = B_0^*\phi\{1 + B_1^*\phi^{\beta^*} + \dots\}, \tag{3.36b}$$

where zeroth- and first-order matching with the inner solution gives

$$A_0^* = A_0, \quad B_0^* = B_0 = (\gamma-1)/\epsilon(\gamma+1), \quad \alpha_0^* = -\epsilon/(k-\sigma-1),$$
$$A_1^* = -\epsilon B_1^*/\alpha_1, \quad B_1^* = b_1(B_2+B_4),$$
$$\alpha_1^* = B_1^* = -\alpha_1/(k-\sigma-1).$$

Here A_0 is unknown.

We now state the important conclusions of the paper. For $k < k_c(\gamma, \sigma)$ in k, σ, γ parameter space we can derive a large time solution to our problem which asymptotically, to zeroth order, is associated with a constant shock speed and which can be calculated numerically. The higher-order terms in the expansions are however indeterminate (see appendix). For $k > k_c(\gamma, \sigma)$ we can again derive a large time solution but in this case the asymptotic shock velocity varies as a power of r, namely $r^{(\delta-1)/\delta}$; the parameter $\delta = \delta(k, \gamma, \sigma)$ can also be computed. Proceeding with the asymptotic analysis in this case, it is apparent that again the solution is to a certain degree indeterminate but for reasons different from and less well known than those for $k < k_c$.

Clearly the critical value k_c is an important parameter in our problem. It separates parameter space into two distinct regions: one, where $k < k_c$, in which the asymptotic shock velocity is constant and one, where $k > k_c$, in which it varies as a power of distance. The value of k_c can be computed numerically as a function of γ and σ.

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Appendix

Here we give details of the higher-order analysis of the asymptotic expansions for $k < k_c$. We start off by writing down the equations satisfied by P_1, U_1 and S_1 in (3.9). These are

$$S_0 U_1'\left(\frac{\gamma+1}{\gamma-1} S_0 - \phi_1\right) - U_0 \phi_1 S_1' + \left(\frac{\beta_1+\sigma-k}{\sigma-k+1} S_0 - \phi_1 S_0'\right) U_1 + \left(\frac{2(\gamma+1)}{\gamma-1} + \frac{\beta_1+\sigma-k}{\sigma-k+1} U_0 - \phi_1 U_0'\right) S_1' = 0, \quad (\text{A } 1a)$$

$$-\phi_1 S_0 U_0 U_1' + \left(\frac{S_0(\gamma+1)}{2} - \frac{\gamma-1}{2}\phi_1\right) P_1' + \left(\frac{\beta_1}{\sigma-k+1} S_0 U_0 - \phi_1 S_0 U_0'\right) U_1 + \left(\frac{\gamma+1}{2} P_0' - \phi_1 U_0 U_0'\right) S_1 + \frac{(\gamma-1)(\beta_1-k)}{2(\sigma+1-k)} P_1 = 0, \quad (\text{A } 1b)$$

$$P_1 = P_0(\gamma S_1/S_0 + 2\phi_1^{\beta_1/(\sigma-k+1)}). \quad (\text{A } 1c)$$

The boundary conditions at the shock are

$$S_1(1) = 0, \quad U_1(1) = a_0, \quad P_1(1) = 2a_0^2. \quad (\text{A } 2)$$

Matching requires the behaviour of P_i, U_i and S_i near $\phi_1 = 0$. For the zeroth-order terms we have

$$U_0 = 1 + \frac{(k-\sigma\gamma)(\gamma-1)}{\{(\sigma+1)\gamma-k\}(\gamma+1)}\left(\frac{a_0^2}{A_0}\right)^{1/\gamma} \phi_1^{(\sigma+1-k/\gamma)/\sigma+1-k} + \dots, \quad (\text{A } 3a)$$

$$P_0 = A_0 + \frac{k(\gamma-1)A_0}{(\gamma+1)(\sigma+1-k/\gamma)}\left(\frac{a_0^2}{A_0}\right)^{1/\gamma} \phi_1^{(\sigma+1-k/\gamma)/\sigma+1-k} + \dots, \quad (\text{A } 3b)$$

with a corresponding expression for S_0. As far as the first-order solution is concerned we observe that (A $1a$–c) are linear and that the solution for each variable consists of a particular integral and a complementary function which contains two arbitrary constants. The particular integral comes from the $\phi_1^{\beta_1/(\sigma-k+1)}$ term in (A $1c$) and, as we shall see, automatically matches with the outer solution. On the other hand one of the arbitrary constants will provide us with an eigenvalue problem for the exponent β_1. Thus the behaviour of U_1 and P_1 near $\phi_1 = 0$ can be written as

$$U_1 = u^* + \frac{2(k-\sigma\gamma)(\gamma-1)}{\gamma\{(\sigma+1)\gamma-k+\beta_1\gamma\}(\gamma+1)}\left(\frac{a_0^2}{A_0}\right)^{1/\gamma}\phi_1^{(\sigma+1-k/\gamma+\beta_1)/\sigma+1-k}+\ldots, \qquad \text{(A 4)}$$

$$P_1 = p^* + \frac{2kA_0(\gamma-1)}{\gamma(\gamma+1)(\sigma+1-k/\gamma+\beta_1)}\left(\frac{a_0^2}{A_0}\right)^{1/\gamma}\phi_1^{(\sigma+1-k/\gamma+\beta_1)/\sigma+1-k}+\ldots,$$

together with an appropriate expansion for R_1. Here u^* and p^*, the leading terms of the complementary function, are undetermined as yet, while the second terms are the leading ones of the particular integral. Depending on the value of β_1, either can dominate. For certain values of β_1, i.e. $\beta_1 = -n(\sigma+1-k/\gamma)$, $n = 1, 2, \ldots$, logarithms may occur in (A 4).

To complete the matching we write the outer expansion (3.4) in inner variables using (3.6). This gives

$$\begin{aligned} u = 1 &+ \frac{(k/\gamma-\sigma)(\gamma-1)}{(\gamma+1)(\sigma+1-k/\gamma)}\left(\frac{a_0^2}{\alpha_0}\right)^{1/\gamma}\phi_1^{(\sigma+1-k/\gamma)/\sigma+1-k}+\ldots,\\ &+ r^{\beta_1}\left\{\frac{2b_1(k/\gamma-\sigma)(\gamma-1)}{\gamma(\gamma+1)(\sigma+1-k/\gamma+\beta_1)}\left(\frac{a_0^2}{\alpha_0}\right)^{1/\gamma}\phi_1^{(\sigma+1-k/\gamma+\beta_1)/\sigma+1-k}+\ldots\right\}\\ &+ \frac{I^*(k/\gamma-\sigma)(\gamma-1)}{(\gamma+1)r^{\sigma+1-k/\gamma}}\left(\frac{a_0^2}{\alpha_0}\right)^{1/\gamma}+\ldots \end{aligned} \qquad \text{(A 5)}$$

and

$$\begin{aligned} P = \alpha_0 &+ \frac{k\alpha_0(\gamma-1)}{(\gamma+1)(\sigma+1-k/\gamma)}\left(\frac{a_0^2}{\alpha_0}\right)^{1/\gamma}\phi_1^{(\sigma+1-k/\gamma)/\sigma+1-k}+\ldots\\ &+ \alpha_1 h(r) + r^{\beta_1}\left\{\frac{2b_1k\alpha_0(\gamma-1)}{\gamma(\gamma+1)(\sigma+1-k/\gamma+\beta_1)}\left(\frac{a_0^2}{\alpha_0}\right)\phi_1^{(\sigma+1-k/\gamma+\beta_1)/\sigma+1-k}+\ldots\right.\\ &+ \left\{\alpha_2 + \frac{k\alpha_0 I^*(\gamma-1)}{(\gamma+1)}\left(\frac{a_0^2}{\alpha_0}\right)^{1/\gamma}\right\}\frac{1}{r^{\sigma+1-k/\gamma}}. \end{aligned} \qquad \text{(A 6)}$$

On matching the zeroth-order terms of P with P_0 [see (A 4)], we conclude that $A_0 = \alpha_0$ and the full zeroth-order problem matches automatically. In fact the zeroth-order inner problem reduces to (3.10 *a–c*) with the boundary condition (3.10 *d*) at the shock and the matching condition $U_0(0) = 1$. This is a particularly simple eigenvalue problem for a_0, for by a change of variable a_0 can be found by a single numerical integration which in addition gives the value of α_0; this of course assumes the existence of a solution to (3.10), an important point which we have dealt with in § 3.2. Looking further at (A 5) we see that if the term $O\{r^{-(\sigma+1-k/\gamma)}\}$ is to match then the appropriate value of β_1 is $\sigma+1-k/\gamma$. But this is a value for which logarithms appear in (3.6) and in that event more singular terms would appear in (A 5) and (A 6). The conclusion is that a successful match cannot be made in this case and I^* must be equated to zero; we also deduce that α_2 is zero.

We can now proceed with the matching of the $O(r^{\beta_1})$ terms. First, matching P_1, we must have $h(r) = r^{\beta_1}$ and $\alpha_1 = p^*$, to be determined from the eigenvalue problem. Clearly the leading term in the expansion of the particular integral in (A 4) for P_1 now matches automatically with the appropriate terms in (A 6). Next we must have $u^* = 0$, for if there were such a term in (A 5) it would manifest itself in the outer expansion as a term in r^{β_1}. But this cannot happen because if such a term were included in the outer expansion (3.4), it would simply be a constant multiple of r^{β_1}, and because of the boundary condition on u at $\phi = 1$, would have to be zero; hence $u^* = 0$. This condition completes the eigenvalue problem for the unknown exponent β_1; this consists of equations (A 1 a–c), the boundary conditions (A 2) and the condition $u^* = 0$. The determination of b_1 cannot be made by the asymptotic analysis and we must conclude that it is dependent on conditions at finite r. It is not difficult to see that a condition on the finite part of an infinite integral will arise at each successive stage of the matching process and so we shall have an infinite number of such conditions.

Stewartson & Thompson (1970) made an attack on the eigenvalue problem posed by their problem, but there, as here, the difficulties seem extreme. For a glimpse of these the reader is referred to their paper.

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